Abstract nonsense

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March 23, 2024

Motivation

Lots of mathematical theories capturing various things

Topology

- Spaces
- Continuous functions

Group theory

- Symmetries of objects
- Symmetry preserving functions

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Thesis

Category theory as a framework for mathematics

Outline

Categories

A *category* consists of...

A collection of *objects*.

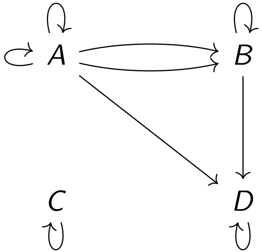
$$A \qquad\qquad B$$

$$C \qquad\qquad D$$

Categories

A *category* consists of...

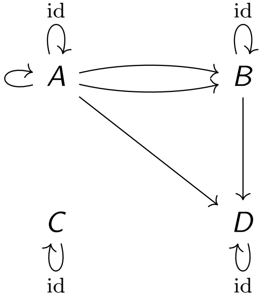
A collection of *morphisms*.



Categories

A *category* consists of...

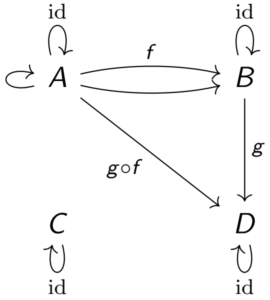
A specified *identity* morphism for each object.



Categories

A *category* consists of...

A specified *composite* morphism for any two composable morphisms.



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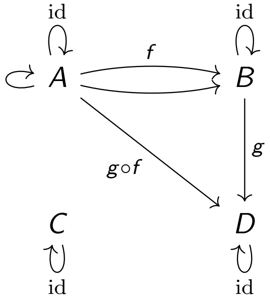


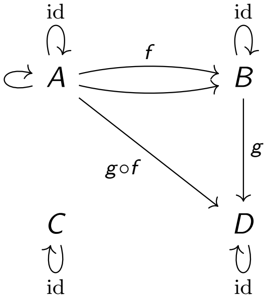
These data are subject to the following requirements:

- **Associativity**: $f \circ (g \circ h) = (f \circ g) \circ h$.
- **Unitality**: $\mathrm{id} \circ f = f = f \circ \mathrm{id}$.

Examples

Set

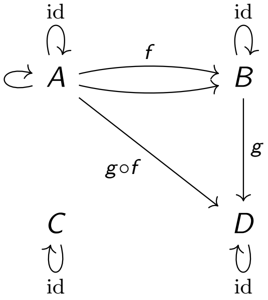
- Objects are sets
- Morphisms are functions
- Identity morphisms are identity functions
- Composition is function composition



Examples

Grp

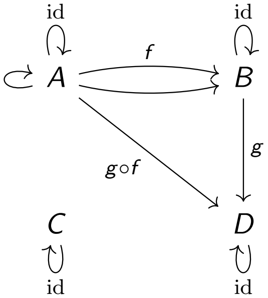
- Objects are groups
- Morphisms are group homomorphisms
- Identity morphisms are identity functions
- Composition is function composition



Examples

Top

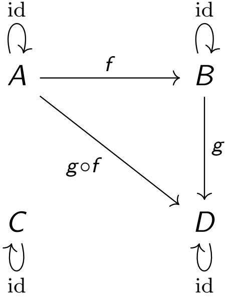
- Objects are topological spaces
- Morphisms are continuous functions
- Identity morphisms are identity functions
- Composition is function composition



Examples

$(\mathrm{P}, \leq)$

- Objects are elements of P
- A morphism $A \to B$ represents the fact that $A \leq B$
- Identity morphism is the reflexivity of $\leq$: $A \leq A$ for any element A
- Composition is the transitivity of $\leq$: $A \leq B$ and $B \leq C$ implies $A \leq C$



Example

Internal groups

Recall the usual presentation of the theory of groups. To specify a group structure on an object G (an *internal group*) is to specify the following data.

- The group identity: $e : 1 \to G$
- The group inverse: $()^{-1} : G \to G$
- The group multiplication: $m : G \times G \to G$

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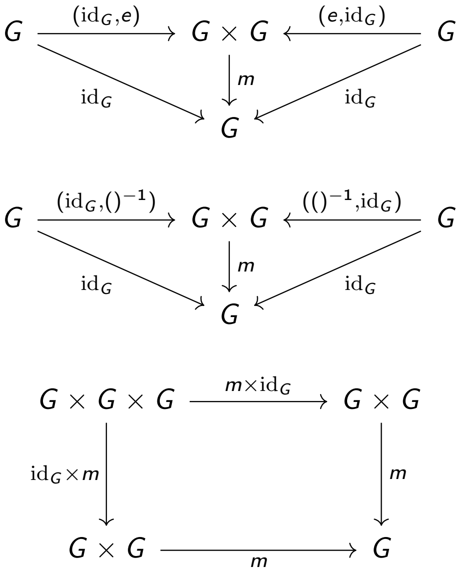
These data are required to satisfy the group axioms.

- $m(x, e) = x = m(e, x)$
- $m(x, x^{-1}) = e = m(x^{-1}, x)$
- $m(m(x, y), z) = m(x, m(y, z))$

Example



G
(id_G, e)
G × G
(e, id_G)
G
id_G
m
id_G
G
G
(id_G, ()^{-1})
G × G
(()^{-1}, id_G)
G
id_G
m
id_G
G
G × G × G
m × id_G
G × G
id_G × m
m
G × G
m
G

Example

Internal groups

An internal group in Set consists of a *set* A and 3 *functions*

- group identity $e : 1 \to A$
- group inverse $()^{-1} : A \to A$
- group multiplication $m : A \times A \to A$

satisfying the group axioms.

An internal group in Top consists of a *topological space* A and 3 *continuous functions*

- group identity $e : 1 \to A$
- group inverse $()^{-1} : A \to A$
- group multiplication $m : A \times A \to A$

satisfying the group axioms.

Observation

An internal group in Set is a group in the usual sense.

Outline

Functors

Let C and D be categories. A *functor* $F : \mathsf{C} \to \mathsf{D}$ consists of the following data.

- An *action on objects*: each object of C is mapped to an object of D
- An *action on morphisms*: each morphism $c \to c'$ is mapped to a morphism $Fc \to Fc'$

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  - Suppressed throughout this talk, but it is an **essential** piece of data of a functor

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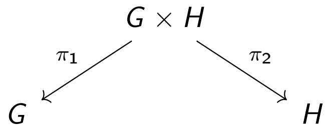
- The $\mathsf{Grp}(-, J) : \mathsf{Grp} \to \mathsf{Set}$ mapping each group I to the set of all group homomorphisms $I \to J$.

Outline

Universality

Motivating examples

We know what products look like in Set. We can generalize its definition to other categories (e.g., Grp). Let G and H be groups. Their *product* $G \times H$ is a group equipped with 2 group homomorphisms π_1 and π_2.



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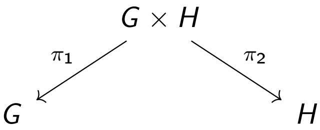
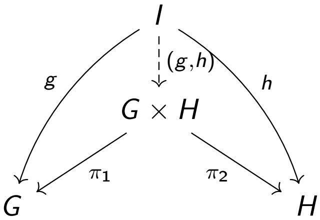
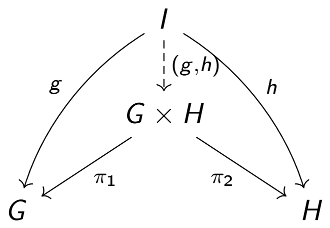


We want this solution to be (1) *general* and (2) *efficient*.

1. For any element g of G and any element h of H, there is an element (g, h) of $G \times H$ such that $\pi_1(g, h) = g$ and $\pi_2(g, h) = h$
2. For any element f of $G \times H$, if $\pi_1 f = g$ and $\pi_2 f = h$ then $f = (g, h)$

We want this solution to be (1) *general* and (2) *efficient*, expressed diagrammatically:





Observation

The generality and efficiency conditions can be encoded as an isomorphism (“natural” in I).

$$\mathrm{Grp}(I, G \times H) \cong \mathrm{Grp}(I, G) \times \mathrm{Grp}(I, H)$$

Representable functors

A set-valued functor $F : \mathsf{C} \to \mathsf{Set}$ is *representable* if there is an object $c \in \mathsf{C}$ and a (natural) isomorphism

$$\mathsf{C}(-, c) \cong F-$$

or

$$\mathsf{C}(c, -) \cong F-$$

The object c is called a *representing* object.

Example: coproducts

In Set, we know what disjoint unions (coproducts) look like. We can generalize this to other categories. The coproduct of two groups G and H is the representing object of the functor $\mathrm{Grp}(G, -) \uplus \mathrm{Grp}(H, -)$ mapping each group I to the disjoint union of the set of all group homomorphisms $G \to I$ and the set of all group homomorphisms $H \to I$. This is represented by the free product (confusingly).

$$\mathrm{Grp}(G + H, -) \cong \mathrm{Grp}(G, -) \uplus \mathrm{Grp}(H, -)$$

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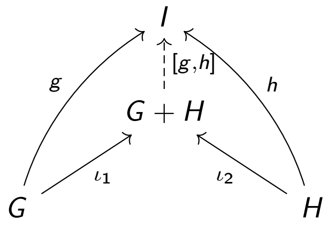
Diagrammatically,



Example: graph coloring

The functor n-Color mapping each graph to the set of all n-colorings is represented by the complete graph K_n.

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- Under this isomorphism, the identity graph homomorphism $\mathrm{id}_{K_n} \in \mathsf{Graph}(K_n, K_n)$ corresponds to an n-coloring in n-Color(K_n). This is called the *universal n-coloring*.

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- Under this isomorphism, the identity graph homomorphism $\mathrm{id}_{K_n} \in \mathsf{Graph}(K_n, K_n)$ corresponds to an n-coloring in n-Color(K_n). This is called the *universal n-coloring*.
- Every graph homomorphism $f : G \to K_n$ determines a unique n-coloring on G by taking the inverse image f^{-1} on the universal n-coloring.

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- Similarly, every subset of S is *classified* by a unique proposition $\varphi : S \to \mathbb{B}$.
- The subset classified by $\varphi : S \to \mathbb{B}$ has a simple description:

$$\{\, s \in S \mid \varphi(s) = t \,\}$$

Suggested reading

Tom Leinster.
Basic category theory.
Cambridge University Press, 2014.

E. Riehl.
Category theory in context.
Aurora: Dover modern math originals. Dover Publications, 2017.