# ELEMENTARY TOPOS

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## Contents



### 1. INTRODUCTION

This personal study note closely follows [Joh02] and [Joh77]. The ultimate goal is to study Diaconescu's theorem, which is applicable in categorical logic.

### 2. BASIC THEORY

**Definition 2.1.** An *(elementary) topos* is a cartesian closed category equipped with a subobject classifier.

**Theorem 2.2.** *Any Grothendieck topos $\mathcal{E}$ is an elementary topos.*

*Proof.* By Giraud's theorem, every Grothendieck topos is cartesian. To define exponentials, we use the Yoneda lemma.

$$B^A(U) \cong \mathcal{E}(\mathsf{C}(-,U), B^A)$$

Taking $B^A(U) := \mathcal{E}(\mathsf{C}(-,U) \times A, B)$, we need to verify that $\mathcal{E}(C \times A, B) \cong \mathcal{E}(C, B^A)$ for all functors $C$. By the density theorem, every functor is a colimit of representables. Thus,

$$\begin{aligned}
\mathcal{E}(C \times A, B) &\cong \mathcal{E}(\mathrm{colim}_U\, \mathsf{C}(-,U) \times A, B) && (1)\\
&\cong \mathcal{E}(\mathrm{colim}_U (\mathsf{C}(-,U) \times A), B) && (2)\\
&\cong \lim_{U^{\mathrm{op}}} \mathcal{E}(\mathsf{C}(-,U) \times A, B) && (3)\\
&\cong \lim_{U^{\mathrm{op}}} \mathcal{E}(\mathsf{C}(-,U), B^A) && (4)\\
&\cong \mathcal{E}(C, B^A) && (5)
\end{aligned}$$

If $B$ is a $J$-sheaf and $A$ is any functor, then since $\mathrm{Sh}(\mathsf{C}, J)$ is an exponential ideal, $B^A$ is a sheaf. Finally, we use the same thought experiment to show that $\mathcal{E}$ has a subobject classifier. Again, by the Yoneda lemma,

$$\Omega(U) \cong \mathcal{E}(\mathsf{C}(-, U), \Omega)$$

A morphism $\mathsf{C}(-, U) \to \Omega$ is required to classify a subfunctor of $\mathsf{C}(-, U)$, or equivalently a sieve on $U$. Thus, we take $\Omega(U)$ to be the set of all sieves on $U$. Note that $\Omega$ just defined is a $J$-sheaf because for any covering sieve $R$ on $U$ and any morphism $R \to F$ the unique extension is the morphism $\mathsf{C}(-, U) \to \Omega$ corresponding to the element $R \in \Omega(U)$ under the isomorphism. □

**Example 2.3.** The category $\mathsf{Set}_f$ of finite sets and functions between them is an elementary topos, but not a Grothendieck topos since it is not cocomplete.

There are two notions of morphisms between toposes. The obvious one preserves all structures of a topos.

**Definition 2.4.** Let $\mathcal{E}$ and $\mathcal{F}$ be topoi. A *logical functor* $F : \mathcal{E} \to \mathcal{F}$ is a functor preserving finite limits, exponentials, and the subobject classifier.

The other one is inspired by geometry.

**Definition 2.5.** Let $\mathcal{E}$ and $\mathcal{F}$ be topoi. A *geometric morphism* $f : \mathcal{E} \to \mathcal{F}$ is an adjoint pair of functors

$$\mathcal{E} \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{F}$$

such that $f^*$ preserves finite limits. The functor $f_*$ is called the *direct image functor* and the functor $f^*$ is called the *inverse image functor*. A geometric morphism is *essential* if the inverse image functor $f^*$ admits a left adjoint $f_!$.

**Definition 2.6.** Let $f, g : \mathcal{E} \to \mathcal{F}$ be a geometric morphisms. A *geometric transformation* $f \Rightarrow g$ is a natural transformation $f_* \Rightarrow g_*$. This transposes to a unique natural transformation $f^* \Rightarrow g^*$ under the adjunction.

**Remark 2.7.** Topoi, geometric morphisms, and geometric transformations assemble into a 2-category $\mathsf{Topoi}$.

**Example 2.8.** Let $f : \mathsf{C} \to \mathsf{D}$ be a functor between small categories. The functor $f^*$ defined by precomposition has a left and right adjoints. The right Kan extension is the direct image, while $f^*$ is the inverse image.

$$\mathrm{Psh}(\mathsf{D}) \underset{\mathrm{Ran}_f}{\overset{f^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Psh}(\mathsf{C})$$

Thus, the assignment $\mathsf{C} \mapsto \mathrm{Psh}(\mathsf{C})$ can be extended to a 2-functor $\mathsf{Cat} \to \mathsf{Topoi}$.

**Lemma 2.9.** *Let* $\mathsf{C}$ *and* $\mathsf{D}$ *be small categories such that* $\mathsf{D}$ *is Cauchy-complete. Then every essential geometric morphism* $f : \mathrm{Psh}(\mathsf{C}) \to \mathrm{Psh}(\mathsf{D})$ *is induced by a functor* $\mathsf{C} \to \mathsf{D}$.

*Proof.* Let $g : \mathsf{C} \to \mathsf{D}$ be a functor. By Example 2.8, this functor induces an essential geometric morphism defined by precomposition.

Conversely, let $f$ be an essential geometric morphism. We want to show that $f$ is induced by a functor $g : \mathsf{C} \to \mathsf{D}$ by precomposition. Note that representable functors are tiny in the presheaf category and $f^*$ preserves colimits. Owing to the following natural isomorphism,

$$\mathrm{Psh}(\mathsf{D})(f_!\mathsf{C}(-, c), -) \cong \mathrm{Psh}(\mathsf{C})(\mathsf{C}(-, c), f^*-)$$

we see that $\mathrm{Psh}(\mathsf{D})(f_!\mathsf{C}(-, c), -)$ preserves sifted colimits. Thus, $f_!\mathsf{C}(-, c)$ is perfectly presentable and therefore it is a retract of a representable. Since $\mathsf{D}$ is Cauchy-complete, $f_!\mathsf{C}(-, c)$ is representable. Thus, $f_!$ restricts along the Yoneda embedding to a functor $f_0 : \mathsf{C} \to \mathsf{D}$. Indeed, $f$ is induced by $f_0$.

$$\begin{aligned} f^*(F)(c) &\cong \mathrm{Psh}(\mathsf{C})(\mathsf{C}(-, c), f^*F) && (6) \\ &\cong \mathrm{Psh}(\mathsf{D})(f_!\mathsf{C}(-, c), F) && (7) \\ &\cong \mathrm{Psh}(\mathsf{D})(\mathsf{D}(-, f_0 c), F) && (8) \\ &\cong F f_0 c && (9) \end{aligned}$$

□

**Lemma 2.10.** *In a topos, every monomorphism is an equalizer.*

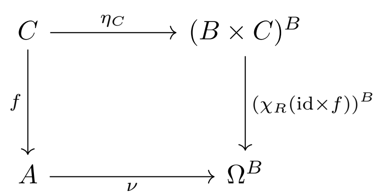
*Proof.* Since $t : 1 \to \Omega$ is a split monomorphism, it is the equalizer of $\mathrm{id}_\Omega$ and $\Omega \xrightarrow{!_\Omega} 1 \xrightarrow{t} \Omega$. Since every monomorphism is a pullback against $t$ in a topos, it follows that every monomorphism is an equalizer. $\square$

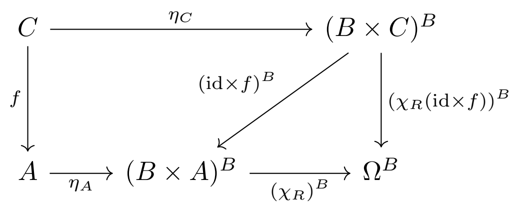
**Corollary 2.11.** *Every topos is balanced.*

Names offer a convenient way to reason about subobjects.

**Lemma 2.12.** *Let $f : C \to A$ be a morphism. If $\nu : A \to \Omega^B$ names a relation $R$ from $B$ to $A$, then $\nu f$ names the relation from $B$ to $C$ obtained by pulling back $R$ along* $\mathrm{id} \times f$.

*Proof.* It suffices to show that $\nu f$ names the same subobject classified by $\chi_R(\mathrm{id} \times f)$, i.e., that the left diagram commutes, where $\eta$ is the unit of the exponential adjunction. This is equivalent, by unfolding the definition of $\nu$, to the right diagram.



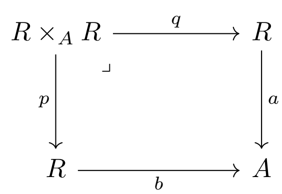


The trapezoid commutes by naturality and the triangle commutes by functoriality. $\square$

### 2.1. Equivalence relations

**Definition 2.13.** Let $(a, b) : R \rightarrowtail A \times A$ be a relation in a category with finite limits.

1. $(a, b)$ is *reflexive* if the diagonal relation $\Delta : A \to A \times A$ factors through $(a, b)$, i.e., there is a map $\rho : A \to R$ such that $a\rho = b\rho = \mathrm{id}_A$.
2. $(a, b)$ is *symmetric* if the twisted relation $(\pi_2, \pi_1)(a, b) : R \to A \times A$ factors through $(a, b)$, i.e., there is a map $\sigma : R \to R$ such that $a\sigma = b$ and $b\sigma = a$.
3. $(a, b)$ is *transitive* if the $R$-related triple relation $(ap, bq) : R \times_A R \to A \times A$ factors through $(a, b)$, i.e., there is a map $\tau : R \times_A R \to R$ such that $a\tau = ap$ and $b\tau = bq$, where $p$ and $q$ are given in the following pullback.

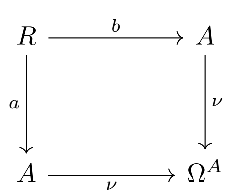


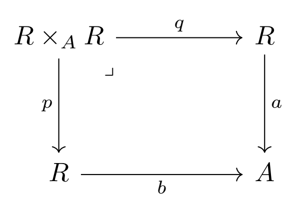
4. $(a, b)$ is an *equivalence relation* if it is reflexive, symmetric, and transitive.

**Definition 2.14.** An equivalence relation is *effective* if it arises as the kernel pair of some morphism.

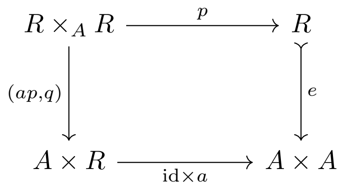
**Theorem 2.15.** *In a topos, equivalence relations are effective.*

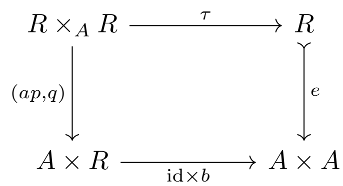
*Proof.* Let $\nu$ be the name of the given equivalence relation $e = (a, b) : R \rightarrowtail A \times A$. We claim that $(a, b)$ is a kernel pair of $\nu$, i.e., the left diagram is a pullback. Let $p$ and $q$ be given as in the right diagram.



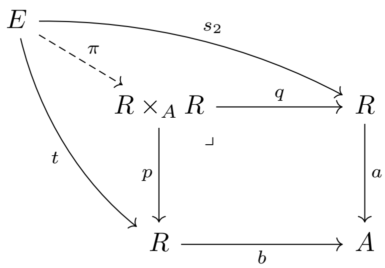


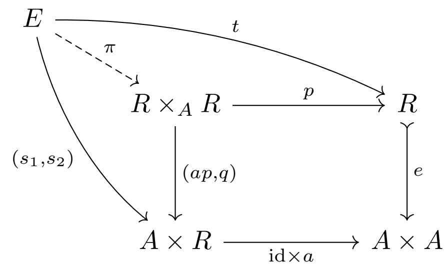
To show that the diagram commutes, it suffices to show that $\nu a$ and $\nu b$ name the same subobject. To this end, it suffices to show that the following two diagrams are pullbacks.



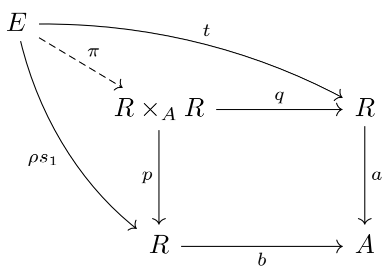



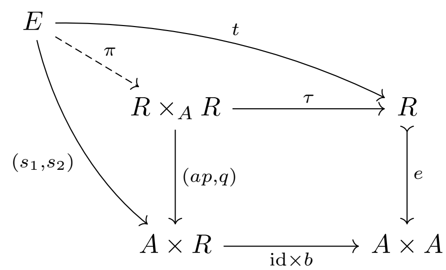

Let $(s_1, s_2) : E \to A \times R$ and $t : E \to R$ with $et = (s_1, as_2)$. Since $ap$ and $q$ are jointly monomorphic, it suffices to show that there is a factorization. Indeed, this factorization is given by the left pullback.



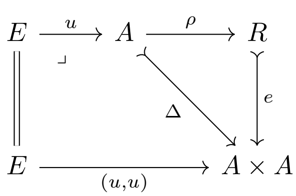



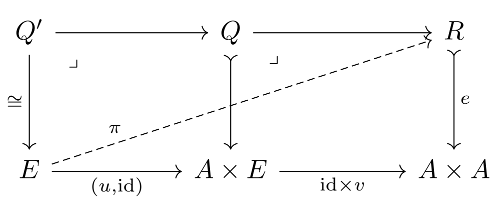

In the second diagram, suppose that $(s_1, s_2) : E \to A \times R$ and $t : E \to R$ with $et = (s_1, bs_2)$. Then a factorization is given by the left pullback.

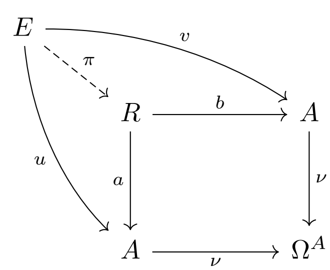



It remains to show that the original square is a pullback. Let $u : E \to A$ and $v : E \to A$ with $\nu u = \nu v$, i.e., the pullbacks of $e$ along $\mathrm{id} \times u$ and $\mathrm{id} \times v$ are the same. Then, pulling both sides back by $(u, \mathrm{id}) : E \to A \times E$ yields the same subobject of $E$. But the pullback of $e$ along $(u, u)$ is the maximal subobject of $E$ because the left diagram is a pullback. This implies that the pullback of $e$ along $(u, v)$ is also the maximal subobject of $E$. Thus, $(u, v)$ factors through $e$ as indicated in the middle diagram. This yields the required factorization.



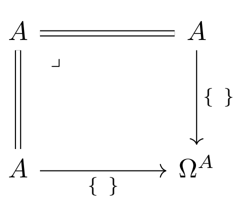




Since $a$ and $b$ are jointly monomorphic, it follows that the factorization is unique. □

**Lemma 2.16.** *Let $\{\ \} : A \to \Omega^A$ be the name of the diagonal equivalence relation $\Delta : A \rightarrowtail A \times A$. $\{\ \}$ is also a mono.*

*Proof.* By Theorem 2.15, the following diagram is a pullback.



Thus, $\{\ \}$ is a monomorphism. $\square$

### 2.2. Powerobjects and pullback functors

For any object $A$ of a topos, we have the *inclusion relation* classified by the evaluation morphism (i.e., the counit of the exponentiation adjunction). And for any morphism $f : B \to A$, we obtain a relation $R$ as follows.

$$\begin{array}{ccc} \in_A & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow t \\ A \times \Omega^A & \xrightarrow[\mathrm{ev}]{} & \Omega \end{array} \qquad\qquad \begin{array}{ccccc} R & \longrightarrow & \in_A & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow t \\ B \times \Omega^A & \xrightarrow[f \times \mathrm{id}]{} & A \times \Omega^A & \xrightarrow[\mathrm{ev}]{} & \Omega \end{array}$$

**Example 2.17.** In $\mathsf{Set}$, the relation $R$ is given by $b\ R\ S \iff f(b) \in S$. Pulling this relation back along $\mathrm{id} \times \{\ \}$ yields a relation from $B$ to $A$ such that $b\ G\ a \iff f(b) \in \{ a \}$. That is, $G$ is the graph of the given function $f$.

$$\begin{array}{ccccc} G & \longrightarrow & R & \longrightarrow & \in_A \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ B \times A & \xrightarrow[\mathrm{id} \times \{\ \}]{} & B \times \Omega^A & \xrightarrow[f \times \mathrm{id}]{} & A \times \Omega^A \end{array}$$

**Lemma 2.18.** *Let $\mathcal{E}$ be a topos. The assignment $A \mapsto \Omega^A$ can be extended to a faithful functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$.*

*Proof.* Given $f : A \to B$, let $Pf : \Omega^B \to \Omega^A$ be the exponential transpose of the following composite. Thus, it follows, by uniqueness, that this assignment is functorial.

$$A \times \Omega^B \xrightarrow{f \times \mathrm{id}} B \times \Omega^B \xrightarrow{\mathrm{ev}} \Omega$$

To show faithfulness, suppose that $Pf = Pg$. Then $Pf\{\ \} = Pg\{\ \}$. These morphisms name the relations $(f, \mathrm{id})$ and $(g, \mathrm{id})$, respectively. Since they are the same relation, it follows that $f = g$. $\square$

**Lemma 2.19.** *In any topos, the powerobject functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ admits a left adjoint.*

*Proof.* The left adjoint is in fact the same functor in the opposite category $P^{\mathrm{op}} : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}$.

$$\frac{\dfrac{\dfrac{\dfrac{\Omega^X \to Y \ (\text{in } \mathcal{E}^{\mathrm{op}})}{Y \to \Omega^X \ (\text{in } \mathcal{E})}}{X \times Y \to \Omega}}{Y \times X \to \Omega}}{X \to \Omega^Y}$$

$\square$

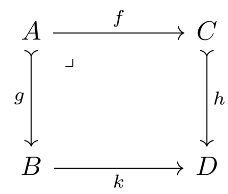
**Example 2.20.** In $\mathsf{Set}$, the powerobject functor constructed in Lemma 2.18 is the usual contravariant powerset functor.

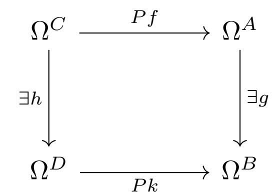
When $f : A \rightarrowtail B$ is a monomorphism, then the composite $\in_A \rightarrowtail A \times \Omega^A \xrightarrow{f \times \mathrm{id}} B \times \Omega^A$ is also a monomorphism. Thus, it has a unique classifying map $(\exists f)^\dagger$ in a topos.

**Definition 2.21.** Let $f : A \rightarrowtail B$ be a monomorphism in a topos, then we define $\exists f : \Omega^A \to \Omega^B$ to be the exponential transpose of the classifying map $(\exists f)^\dagger$.

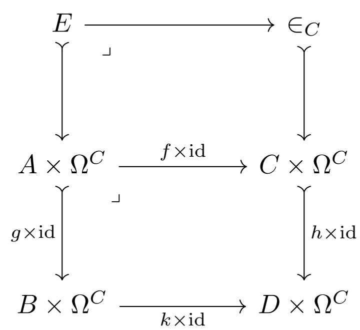
**Example 2.22.** Given an injective function $f : A \to B$ in $\mathsf{Set}$, the composite $\in_A \rightarrowtail A \times \Omega^A \xrightarrow{f \times \mathrm{id}} B \times \Omega^A$ is the relation $f(a)\ R\ S \iff a \in_A S$. This is classified by the truth-valued function $(\exists f)^\dagger : B \times \Omega^A \to \Omega$ that maps $(b, S)$ to $t$ if and only if $\exists a \in A.\, f(a) = b \wedge a \in S$. The exponential transpose maps $S$ to the subset $\{ b \in B \mid \exists a \in A.\, f(a) = b \wedge a \in S \}$.

**Lemma 2.23** (Beck-Chevalley condition)**.** *If the left diagram is a pullback diagram in a topos, then the right diagram commutes.*





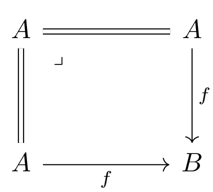

*Proof.* It suffices to show that the two composites $\exists g Pf$ and $Pk\exists h$ name the same subobject. Since $(-) \times \Omega^C$ preserves pullbacks, we have the following pullback diagram.

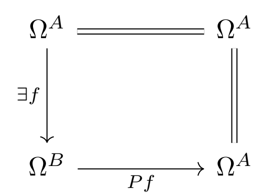



$Pf$ names $E$ as a subobject of $A \times \Omega^C$, so $\exists g Pf$ names the composite $E \rightarrowtail A \times \Omega^C \xrightarrow{g\times \mathrm{id}} B \times \Omega^C$. And $\exists h$ names $\in_C$ as a subobject of $D \times \Omega^C$ whose pullback along $k \times \mathrm{id}$ is $E$. Thus, $Pk\exists h$ also names $E$ as a subobject of $B \times \Omega^C$. □

**Corollary 2.24.** *If $f : A \to B$ is a monomorphism, then $Pf\exists f = \mathrm{id}_{\Omega^A}$.*

*Proof.* Since the left diagram is a pullback, by Beck-Chevalley condition, the right diagram commutes.





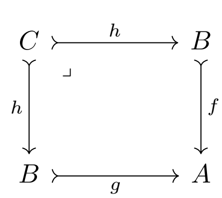

□

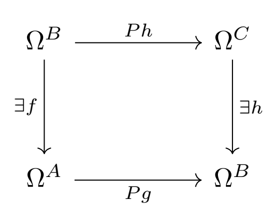
**Theorem 2.25.** *The powerobject functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ of any topos is monadic.*

*Proof.* Since $P$ admits a left adjoint, by the crude monadicity theorem, it suffices to show that $P$ reflects isomorphisms and that $\mathcal{E}^{\mathrm{op}}$ has reflexive coequalizer and $P$ preserves them.

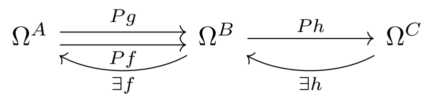
Since $\mathcal{E}$ is finitely complete, $\mathcal{E}^{\mathrm{op}}$ is finitely cocomplete. Thus, $\mathcal{E}^{\mathrm{op}}$ trivially has reflexive coequalizer (because it has all coequalizers). Note that the property of being balanced is self dual, so it suffices to show that $P$ reflects monomorphisms and epimorphisms. This follows immediately from the fact that $P$ is faithful.

It remains to show that $P$ preserves reflexive coequalizers. A reflective coequalizer in $\mathcal{E}^{\mathrm{op}}$ is a coreflective equalizer in $\mathcal{E}$, which can be expressed as the pullback on the left. By Beck-Chevalley condition, the right diagram commutes.





Since $h$ and $g$ are monomorphisms, $Ph\exists h = \mathrm{id}_{\Omega^C}$ and $Pg\exists g = \mathrm{id}_{\Omega^B}$. Thus, the following diagram is a split coequalizer (therefore, a reflexive coequalizer) in $\mathcal{E}$.



□

**Corollary 2.26.** *Every topos $\mathcal{E}$ is finitely cocomplete.*

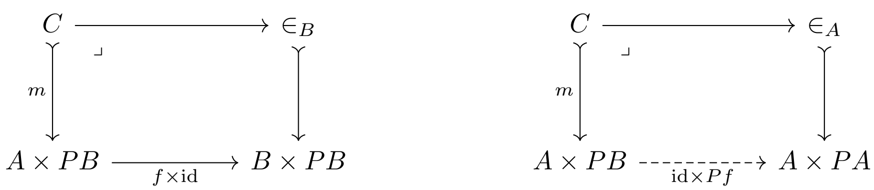
*Proof.* Since the powerobject functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is monadic, it creates limits. Thus, $\mathcal{E}^{\mathrm{op}}$ has finite limits. But finite limits in $\mathcal{E}^{\mathrm{op}}$ are finite colimits in $\mathcal{E}$. □

**Definition 2.27.** Let $\mathsf{C}$ be a cartesian category and $A$ be an object. The *powerobject* $PA$ of $A$ consists of a universal inclusion relation $\in_A \rightarrowtail A \times PA$ such that for any relation $R \rightarrowtail A \times B$, there is a unique comparison map $\mathrm{id} \times f : A \times B \to A \times PA$ whose pullback against $\in_A$ is the relation $R$.

The abuse of notation in Definition 2.27 is justified since, in a topos, $PA$ is a powerobject, where $P$ is the powerobject functor. In fact, topoi can be equivalently reformulated as cartesian categories in which every object has a powerobject. One direction is clear: every topos has powerobjects given by the powerobject functor.

**Lemma 2.28.** *Let $\mathsf{C}$ be a cartesian category with powerobjects. The assignment $A \mapsto PA$ can be extended to a functor $P : \mathsf{C}^{\mathrm{op}} \to \mathsf{C}$.*

*Proof.* Let $f : A \to B$ be a morphism. Take the pullback indicated on the left. Then let $Pf$ be given by the universal property of powerobjects as indicated on the right.



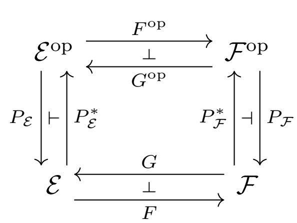
Uniqueness implies functoriality. □

**Remark 2.29.** In a topos, the functor constructed in Lemma 2.28 is just the usual powerobject functor.

**Remark 2.30.** The proof of Theorem 2.25 can be adapted to show that the functor in Lemma 2.28 is monadic. Similarly, we can show that every cartesian category with powerobjects is balanced, providing the analogue of Corollary 2.11.

**Lemma 2.31.** *Let $F : \mathcal{E} \to \mathcal{F}$ be a powerobject-preserving functor between cartesian categories with powerobjects. $F$ admits a left adjoint iff it admits a right adjoint.*

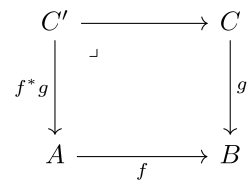
*Proof.* Suppose that $G \dashv F$, then $F^{\mathrm{op}} \dashv G^{\mathrm{op}}$. And since $F$ preserves powerobjects, the following diagram commutes up to isomorphism in both directions. We claim that $P^*_{\mathcal{E}} G P_{\mathcal{F}}$ is left adjoint to $F^{\mathrm{op}}$.



$$\begin{array}{c}
P^*_{\mathcal{E}} G P_{\mathcal{F}} f \to e \quad \text{in } \mathcal{E}^{\mathrm{op}} \\ \hline
G P_{\mathcal{F}} f \to P_{\mathcal{E}} e \quad \text{in } \mathcal{E} \\ \hline
P_{\mathcal{F}} f \to F P_{\mathcal{E}} e \quad \text{in } \mathcal{F} \\ \hline
P_{\mathcal{F}} f \to P_{\mathcal{F}} F^{\mathrm{op}} e \quad \text{in } \mathcal{F} \\ \hline
P^*_{\mathcal{F}} P_{\mathcal{F}} f \to F^{\mathrm{op}} e \quad \text{in } \mathcal{F}^{\mathrm{op}} \\ \hline
f \to F^{\mathrm{op}} e \quad \text{in } \mathcal{F}^{\mathrm{op}}
\end{array}$$

Note that the counit $\epsilon_f$ of the adjunction $P^*_{\mathcal{F}} \dashv P_{\mathcal{F}}$ is a coequalizer of a pair of morphisms (so it is epimorphic) because $P_{\mathcal{F}}$ is monadic. But since $P_{\mathcal{F}}$ is faithful, it is also monomorphic. Thus, it follows that $\epsilon_f$ is an isomorphism. Since $F^{\mathrm{op}}$ admits a left adjoint, $F$ admits a right adjoint. The converse follows analogously. □

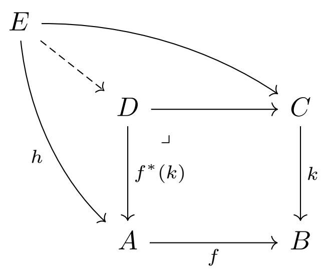
**Notation 2.32.** Let $\mathsf{C}$ be a category with pullbacks and $f : A \to B$ be a morphism, we write $f^*$ for the functor $\mathsf{C}/B \to \mathsf{C}/A$ obtained by pulling back along $f$. The action on morphisms is the canonical one given by the universal property of pullbacks.




We write $B^* : \mathsf{C} \cong \mathsf{C}/1 \to \mathsf{C}/B$ for the pullback functor along the unique map $B \to 1$.

**Lemma 2.33.** *Let $f : A \to B$ be a morphism in a category $\mathsf{C}$ with pullbacks. The pullback functor $f^* : \mathsf{C}/B \to \mathsf{C}/A$ admits a faithful left adjoint $\Sigma_f : \mathsf{C}/A \to \mathsf{C}/B$.*

*Proof.* Consider the forgetful functor $p : \mathsf{C}/A \cong (\mathsf{C}/B)/f \to \mathsf{C}/B$, which is clearly faithful. Indeed, the forgetful functor is given by the assignment $h \mapsto fh$ on objects. Then a morphism $p(h) \to k$ yields a cone in $\mathsf{C}$ as in the left diagram. This corresponds to a unique morphism $h \to f^*(k)$, yielding the required natural bijection.

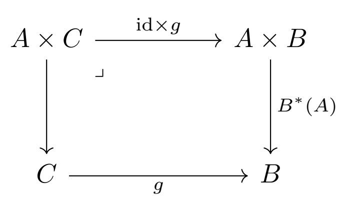


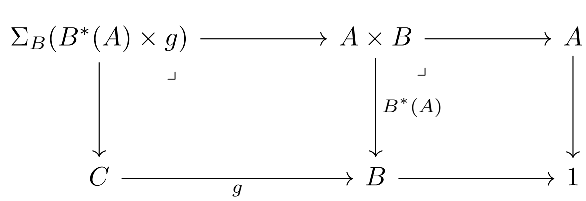

$$\frac{p(h) \to k}{h \to f^*(k)}$$

Thus, we take $\Sigma_f := p$. $\square$

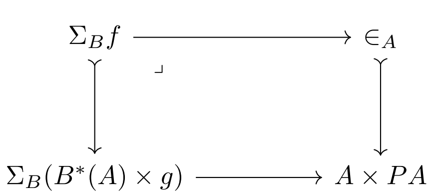
**Lemma 2.34.** *Let $A$ be an object of a cartesian category $\mathsf{C}$ with a powerobject $PA$. Then, for any object $B$ of $\mathsf{C}$, the object $B^*(PA)$, equipped with the relation $B^*(\in_A)$, is a powerobject for $B^*(A)$ in $\mathsf{C}/B$.*

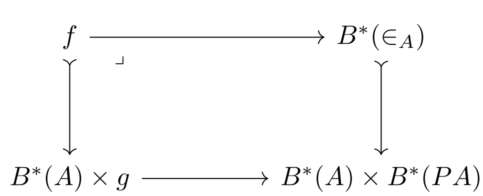
*Proof.* Given any $g : C \to B$, the left diagram is a pullback. Thus, the right diagram implies that $\Sigma_B(B^*(A) \times g) \cong A \times C$.



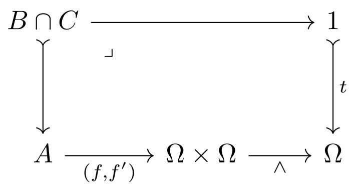

Note that $\Sigma_B$ preserves monomorphisms (because it creates connected limits). Thus, for any $f \rightarrowtail B^*(A) \times g$, we have a relation $\Sigma_B f \rightarrowtail \Sigma_B(B^*(A) \times g)$ and the left diagram. Transposing across the adjunction $\Sigma_B \dashv B^*$ yields the desired pullback.

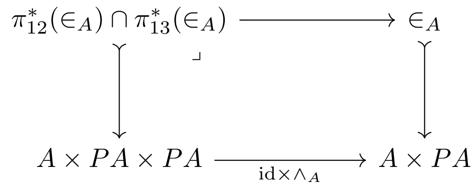


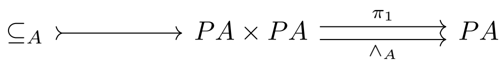

$\square$

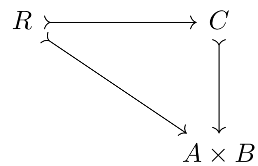
**Example 2.35.** In a cartesian category with powerobjects, the powerobject for the terminal object 1 is a subobject classifier. Thus, we can define $\wedge : \Omega \times \Omega \to \Omega$ to be the classifying map for the subobject $(t, t) : \Omega \rightarrowtail \Omega \times \Omega$. Now, let $f, f' : A \rightrightarrows \Omega$ be a pair of maps classifying $B \rightarrowtail A$ and $C \rightarrowtail A$, respectively. We define $B \cap C$ to be the subobject classified by the composite $\wedge(f, f')$. Consider the subobject $\pi_{12}^*(\in_A) \cap \pi_{13}^*(\in_A)$ of $A \times PA \times PA$. By the universal property of powerobjects, there is a unique name $PA \times PA \to PA$ for this subobject.




In Set, $\wedge_A$ is set intersection. This intuition leads us to consider the following equalizer as the subset relation.



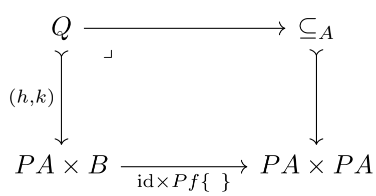




Now, suppose that we have a subobject $R \rightarrowtail C$ and a relation $C \rightarrowtail A \times B$. Then the composite yields a relation $R \rightarrowtail A \times B$. Let $\nu_R$ be the name of $R \rightarrowtail A \times B$ and $\nu_C$ be the name of $C \rightarrowtail A \times B$. Then $\nu_R = \wedge_A(\nu_R, \nu_C)$.

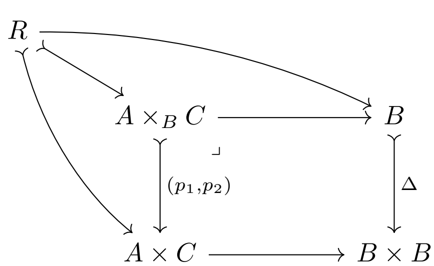
**Theorem 2.36.** *If* $\mathsf{C}$ *is a cartesian category with powerobjects and* $B$ *is an object of* $\mathsf{C}$*, then* $\mathsf{C}/B$ *is also a cartesian category with powerobjects.*

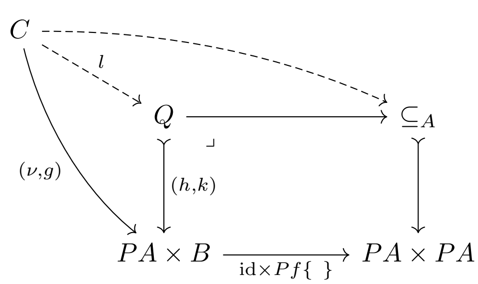
*Proof.* **TODO:** This is my best attempt at deciphering Johnstone's proof. Parts of the proof have not been deciphered.

$\mathsf{C}/B$ is a cartesian category: the terminal object is $\mathrm{id}_B$ and its coequalizers are created by $\Sigma_B$. Let $f : A \to B$ be an arbitrary object in $\mathsf{C}/B$, we need to construct a powerobject for it. In Set, we can view $f$ as an $B$-indexed family whose total space is $A$. Then we expect its powerobject $k$ to be an $B$-indexed family of subsets of fibers of $f$.




For any object $g : C \to B$ in $\mathsf{C}/B$, a subobject of $f \times g$ is a subobject of $A \times_B C$ in $\mathsf{C}$, which corresponds uniquely to a pair of morphisms $R \rightarrowtail A \times C$ and $R \to B$. Since $\Delta$ is monomorphic, the latter is unique. Thus, every subobject of $A \times_B C$ corresponds to a unique subobject of $A \times C$. Let $\nu_R$ be the name of the relation $R \rightarrowtail A \times C$. Since $Pf\{\ \}g$ names $A \times_B C \xrightarrow{(p_1,p_2)} A \times C$, we have $\nu_R = \wedge_A(\nu_R, Pf\{\ \}g)$. This yields a unique cone over the pullback, corresponding to a unique morphism $l : C \to Q$ such that $kl = g$, i.e., a morphism $g \to k$ in $\mathsf{C}/B$.

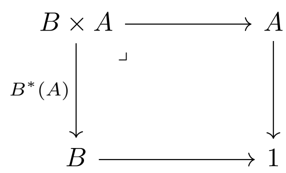


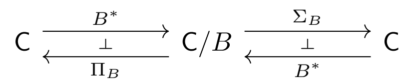




This establishes a bijection between subobjects of $f \times g$ in $\mathsf{C}/B$ and morphisms $g \to k$. We take $\in_f \rightarrowtail f \times k$ to be the one that corresponds to the identity $k \to k$. Then we can readily see that $\in_f \rightarrowtail f \times k$ has the required universal property through the bijection constructed above. ☐

**Corollary 2.37.** *For any object* $B$ *in a cartesian category* $\mathsf{C}$ *with powerobjects,* $B^*$ *admits a right adjoint* $\Pi_B$ *and* $\mathsf{C}$ *is cartesian closed.*

*Proof.* Since $B^*$ is a powerobject-preserving functor admitting a left adjoint, it has a right adjoint by Lemma 2.31. Note that the domain of $B^*(A)$ is the product $B \times A$. Thus, the composite $\Sigma_B B^*$ is the product functor. Then the composite $\Pi_B B^*$ is the required right adjoint.





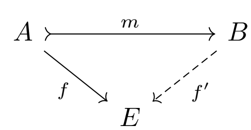
□

**Corollary 2.38.** *Every cartesian category with powerobjects is a topos.*

**Corollary 2.39.** *If $\mathcal{E}$ is a topos and $B$ is any object of $\mathcal{E}$, then $\mathcal{E}/B$ is also a topos.*

### 2.3. Injective objects

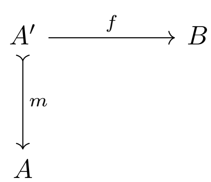
**Definition 2.40.** An object $E$ is *injective* if, for any monomorphism $m : A \rightarrowtail B$, every morphism $f : A \to E$ has a (not necessarily unique) extension $f' : B \to E$.

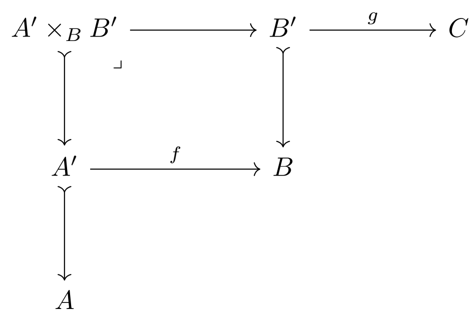


**Definition 2.41.** A category has *enough injectives* if, for any object $A$, there is an injective object $E$ and a monomorphism $A \rightarrowtail E$.

### 2.4. Partial maps and their classifiers

**Definition 2.42.** A *partial map* $f : A \rightharpoonup B$ in a cartesian category $\mathsf{C}$ is a diagram indicated on the left. Partial maps can be composed as indicated on the right.





Up to isomorphism, composition of partial maps is associative and unital. By identifying isomorphic relations, this yields a category $\mathsf{C}_\perp$ whose objects are those of $\mathsf{C}$ and whose morphisms are partial maps of $\mathsf{C}$. There is a functor $p : \mathsf{C} \to \mathsf{C}_\perp$ that is identity on object and sends $f : A \to B$ to the partial map $(\mathrm{id}, f) : A \rightharpoonup B$.

**Definition 2.43.** A cartesian category $\mathsf{C}$ has *representable partial maps* if $p$ admits a right adjoint.

**Theorem 2.44.** *Every topos has representable partial maps.*

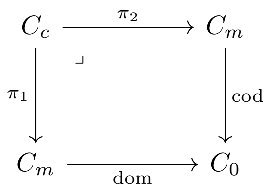
*Proof.* **TODO:** □

## 3. INTERNAL CATEGORIES

Category theory is an essentially algebraic theory. Thus, it can be internalized in any category with finite limits. For the time being, let us assume that $\mathcal{E}$ has finite limits.

**Definition 3.1.** An *internal category* of $\mathcal{E}$ consists of the following data.

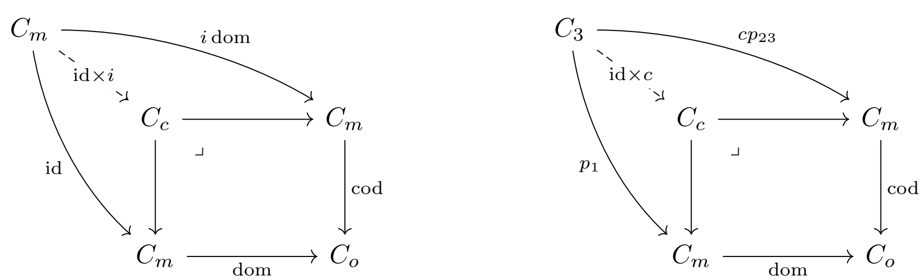
- An object of objects $C_o$ and an object of morphisms $C_m$.
- A domain and a codomain morphism $\mathrm{dom}, \mathrm{cod} : C_m \to C_o$, an identity $i : C_o \to C_m$, and a composition $c : C_c \to C_m$, where $C_c$[1] is the following pullback.




These data are subject to the following requirements.

- $\mathrm{dom}\, i = \mathrm{cod}\, i = \mathrm{id}_{C_o}$.
- $\mathrm{dom}\, c = \mathrm{dom}\, \pi_2$ and $\mathrm{cod}\, c = \mathrm{cod}\, \pi_1$.
- $c(\mathrm{id} \times i) = c(i \times \mathrm{id}) = \mathrm{id}$ and $c(\mathrm{id} \times c) = c(c \times \mathrm{id})$.

Here, $\mathrm{id} \times i$ and $\mathrm{id} \times c$ are given by the following left and right diagrams, respectively, where $C_3$ is the object of composable triples.




The other two morphisms are obtained in the same fashion.

**TODO:** Construct the object of composable triples.

**Definition 3.2.** An *internal functor* $F : \mathsf{C} \to \mathsf{D}$ consists of an action on objects $f_o : C_o \to D_o$ and an action on morphisms $f_m : C_m \to D_m$. These data are subject to the following requirements.

- $\mathrm{dom}_{\mathsf{D}}\, f_m = f_o\, \mathrm{dom}_{\mathsf{C}}$ and $\mathrm{cod}_{\mathsf{D}}\, f_m = f_o\, \mathrm{cod}_{\mathsf{C}}$.
- $f_m i_{\mathsf{C}} = i_{\mathsf{D}} f_o$ and $f_m c_{\mathsf{C}} = c_{\mathsf{D}}(f_m \times f_m)$.

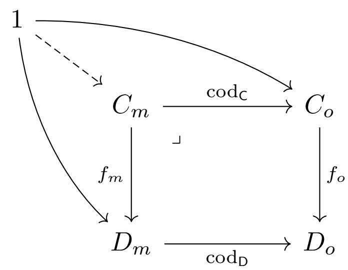
**Definition 3.3.** Let $F : \mathsf{C} \to \mathsf{D}$ be an internal functor. $F$ is a *discrete opfibration* if the left diagram is a pullback. It is a *discrete fibration* if the right diagram is a pullback.


[1] We may think of $C_c$ as the object of composable pairs of morphisms.

**Lemma 3.4.** *Let $f : \mathsf{C} \to \mathsf{D}$ and $g : \mathsf{D} \to \mathsf{E}$ be internal functors such that $g$ is a discrete opfibration (fibration). Then $f$ is a discrete opfibration (fibration) iff $gf$ is.*

*Proof.* Immediate from the pasting lemma of pullbacks. □

**Example 3.5.** Let $\mathsf{C}$ and $\mathsf{D}$ be two internal categories of $\mathsf{Set}$. Consider a discrete fibration $F : \mathsf{C} \to \mathsf{D}$. Let $A$ be an object of $\mathsf{C}$ ($A \in C_o$) and $g : B \to F(A)$ be a morphism in $\mathsf{D}$. Then the universal property of pullbacks provides a unique lift $\widetilde{g} : \widetilde{B} \to A$ of $g$ in $\mathsf{C}$ such that $F(\widetilde{g}) = g$.




**Definition 3.6.** Let $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$ be two internal functors. An *internal natural transformation* $F \Rightarrow G$ consists of an assignment $\alpha : \mathsf{C}_o \to \mathsf{D}_m$ subject to the following requirements.

- $\mathrm{dom}_\mathsf{D}\, \alpha = f_o$ and $\mathrm{cod}_\mathsf{D}\, \alpha = g_o$.
- $c_\mathsf{D}(g_m, \alpha\, \mathrm{dom}_\mathsf{C}) = c_\mathsf{D}(\alpha\, \mathrm{cod}_\mathsf{C}, f_m)$.

**Remark 3.7.** Let $\mathcal{E}$ be a finitely complete category. Internal categories, internal functors, and internal natural transformations assemble into a 2-category $\mathsf{cat}(\mathcal{E})$. In fact, internal categories and internal functors assemble into a 1-category.

**Definition 3.8.** Let $\mathsf{C} \in \mathsf{cat}(\mathcal{E})$.

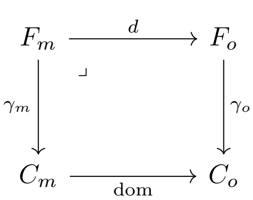
1. $\mathsf{C}$ is a *poset* if dom and cod are jointly monomorphic.
2. $\mathsf{C}$ is *discrete* if $i$ is an isomorphism.
3. The *opposite category* $\mathsf{C}^{\mathrm{op}}$ is obtained by exchanging the roles of dom and cod and composing $c$ with the twist map.

In classical category theory, we are comfortable with set-valued functors from a small category. We can not construct such functors as internal functors because the collection of all sets is not a set (neither is the collection of all functions). But, we can compress the data of a set-valued functor into its category of elements via Grothendieck construction.

**TODO:** Fix the definition and provide an example.

**Definition 3.9.** Let $\mathsf{C} \in \mathsf{cat}(\mathcal{E})$. An *internal diagram* $F$ on $\mathsf{C}$ consists of the following data.

- An object $F_o \xrightarrow{\gamma_o} C_o$ of $\mathcal{E}/C_o$.
- A morphism $e : F_m \to F_o$, where $F_m$ is given by the following pullback.




These data are subject to the following conditions.

- $\gamma_o e = \mathrm{cod}\, \gamma_m$.
- $e(i \times \mathrm{id}) = \mathrm{id}_{F_o}$ and $e(c \times \mathrm{id}) = e(e \times \mathrm{id})$.

A morphism of internal diagrams $F \to G$ is a morphism $F_o \to G_o$ in the category $\mathcal{E}/C_o$. We write $\mathcal{E}^{\mathsf{C}}$ for the category of internal diagrams on $\mathsf{C}$. An internal diagram on $\mathsf{C}^{\mathrm{op}}$ is called an *internal presheaf*.

**Remark 3.10** (Grothendieck construction)**.** Given an internal diagram $F$ on $\mathsf{C}$, we can construct an internal category $\mathsf{F}$ such that $\gamma_o$ and $\gamma_m$ in the pullback diagram of Definition 3.9 form a discrete opfibration $\mathsf{F} \to \mathsf{C}$. Conversely, given a discrete opfibration $\mathsf{G} \to \mathsf{C}$, we readily have an internal diagram on $\mathsf{C}$. Passing this internal diagram through the Grothendieck construction, we obtain the discrete opfibration that we started with. Thus, we have the following equivalence of categories, where $\mathsf{doFib}(\mathsf{C})$ is the full subcategory of $\mathsf{cat}(\mathcal{E})/\mathsf{C}$ spanned by discrete opfibrations.

$$\mathcal{E}^{\mathsf{C}} \simeq \mathsf{doFib}(\mathsf{C})$$

### 3.1. Colimits in internal categories

Although what we are about to develop can be done in a more general category than a topos, we are going to assume that the ambient category $\mathcal{E}$ is a topos from now on.

**Definition 3.11.** Define the *connected component functor* $\pi_0 : \mathsf{cat}(\mathcal{E}) \to \mathcal{E}$ by the following coequalizer.

$$C_m \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\rightrightarrows}} C_o \twoheadrightarrow \pi_0\mathsf{C}$$

**Example 3.12.** Let $\mathsf{C}$ be a small category in the classical sense. Then the functor $\pi_0 : \mathsf{Cat} \to \mathsf{Set}$ maps $\mathsf{C}$ to the set in which objects of $\mathsf{C}$ (elements of $C_o$) connected by a morphism are identified. This is precisely the usual notion of the connected component of a small category.

**Definition 3.13.** Let $\mathsf{C} \in \mathsf{cat}(\mathcal{E})$. Define the *colimit functor* $\mathrm{colim}_{\mathsf{C}} : \mathcal{E}^{\mathsf{C}} \to \mathcal{E}$ by $\mathrm{colim}_{\mathsf{C}} F = \pi_0\mathsf{F}$, where $\mathsf{F}$ is the internal category obtained by passing the internal diagram $F$ through the Grothendieck construction.

**Example 3.14.** Let $\mathsf{C}$ be a small category. Given a set valued diagram $D : \mathsf{C} \to \mathsf{Set}$, the Grothendieck construction yields the category of elements $\int D$ whose connected component is precisely the representing object for the cocone functor.

Every object of $\mathcal{E}$ can be equipped with the structure of an internal category: equip it with the structure of a discrete category. Indeed, this assignment is functorial. We denote this functor by $\Delta : \mathcal{E} \to \mathsf{cat}(\mathcal{E})$.

**Lemma 3.15.** *The functor $\pi_0 : \mathsf{cat}(\mathcal{E}) \to \mathcal{E}$ is left adjoint to $\Delta$.*

*Proof.* Every morphism $\pi_0\mathsf{C} \to D$ in $\mathcal{E}$ corresponds to a unique morphism $C_o \xrightarrow{g} D$ such that $g\,\mathrm{dom} = g\,\mathrm{cod}$. But this corresponds to a unique internal functor from $\mathsf{C}$ to the discrete category $\Delta D$.

$$\begin{array}{ccc} C_m & \xrightarrow{\mathrm{dom}} & C_o \\ {\scriptstyle g\,\mathrm{dom}}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[\mathrm{id}]{} & D \end{array} \qquad\qquad \begin{array}{ccc} C_m & \xrightarrow{\mathrm{cod}} & C_o \\ {\scriptstyle g\,\mathrm{dom}}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[\mathrm{id}]{} & D \end{array}$$

□

**Lemma 3.16.** $\mathrm{colim}_{\mathsf{C}} : \mathcal{E}^{\mathsf{C}} \to \mathcal{E}$ *admits a right adjoint.*

*Proof.* Consider the following adjoints.

$$\mathsf{cat}(\mathcal{E})/\mathsf{C} \underset{\mathsf{C}^*}{\overset{\Sigma_{\mathsf{C}}}{\rightleftarrows}} \mathsf{cat}(\mathcal{E}) \underset{\Delta}{\overset{\pi_0}{\rightleftarrows}} \mathcal{E}$$

Since $\mathsf{C}^*$ sends each internal category $\mathsf{D}$ to the discrete opfibration $\mathsf{C} \times \mathsf{D} \xrightarrow{\pi_1} \mathsf{C}$, we may restrict the codomain of $\mathsf{C}^*$ to discrete opfibrations. Then we may apply the equivalence $\mathcal{E}^{\mathsf{C}} \simeq \mathsf{doFib}(\mathsf{C})$.

$$\mathsf{doFib}(\mathsf{C}) \underset{\mathsf{C}^*}{\overset{\Sigma_{\mathsf{C}}}{\rightleftarrows}} \mathsf{cat}(\mathcal{E}) \underset{\Delta}{\overset{\pi_0}{\rightleftarrows}} \mathcal{E} \qquad\qquad \mathcal{E}^{\mathsf{C}} \underset{\mathsf{C}^*}{\overset{\Sigma_{\mathsf{C}}}{\rightleftarrows}} \mathsf{cat}(\mathcal{E}) \underset{\Delta}{\overset{\pi_0}{\rightleftarrows}} \mathcal{E}$$

But in this case, $\Sigma_{\mathsf{C}}$ is just the Grothendieck construction. Thus, $\pi_0\Sigma_{\mathsf{C}}$ is the colimit functor and $\mathsf{C}^*\Delta$ is the desired right adjoint. □

**Example 3.17.** In $\mathsf{Set}$, the functor $\Delta : \mathsf{Set} \to \mathsf{Cat}$ sends each set $D$ to the discrete category $\mathsf{D}$ containing that set. Then $\mathsf{C}^*$ sends this discrete category to the discrete opfibration $\mathsf{C} \times \mathsf{D} \xrightarrow{\pi_1} \mathsf{C}$, which corresponds to the constant diagram mapping each object of $\mathsf{C}$ to the set $D$. Indeed, $\mathsf{C} \times \mathsf{D}$ is the category of elements of this diagram and $\pi_1$ is the associated forgetful functor.

**Remark 3.18.** The external and the internal notions of cocompleteness are not the same. By Lemma 3.16, $\mathsf{Set}_f$ is internally cocomplete. But, $\mathsf{Set}_f$ is not a cocomplete category.

### 3.2. Filtered categories

Recall that a small category $\mathsf{C}$ is filtered if the following conditions hold:

- $\mathsf{C}$ is nonempty.
- For any pair of objects in $\mathsf{C}$, there is a pair of morphisms with a common codomain.
- For any parallel pair of morphisms, there is a coequalizing cocone.

To translate this definition into a topos, we define the following objects. Let $\mathsf{C} \in \mathsf{cat}(\mathcal{E})$ in the following definitions.

**Definition 3.19.** The *object of morphisms with common codomain* $P_{\mathsf{C}}$, the *object of parallel pairs* $R_{\mathsf{C}}$, and the *object of coequalizing cocones* $T_{\mathsf{C}}$ are respectively defined by the following pullbacks.

$$
\begin{array}{ccc}
P_{\mathsf{C}} & \xrightarrow{p_2} & C_m \\
{\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{cod}} \\
C_m & \xrightarrow[\mathrm{cod}]{} & C_o
\end{array}
\qquad
\begin{array}{ccc}
R_{\mathsf{C}} & \longrightarrow & C_m \\
\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{dom},\mathrm{cod})} \\
C_m & \xrightarrow[(\mathrm{dom},\mathrm{cod})]{} & C_o \times C_o
\end{array}
\qquad
\begin{array}{ccc}
T_{\mathsf{C}} & \longrightarrow & C_c \\
\downarrow & \lrcorner & \downarrow{\scriptstyle (\pi_1,c)} \\
C_c & \xrightarrow[(\pi_1,c)]{} & C_m \times C_m
\end{array}
$$

**Definition 3.20.** An internal category $\mathsf{C}$ is *filtered* if the morphisms $C_o \to 1$, $P_{\mathsf{C}} \to C_o \times C_o$, and $T_{\mathsf{C}} \to R_{\mathsf{C}}$ are epimorphic.

## 4. THE 2-CATEGORY $\mathsf{Topoi}$

## Bibliography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