SET THEORY

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1. INTRODUCTION

Set theory appeared in Cantor's 1874 paper: „Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen." Later, Bertrand Russell discovered a contradiction in Cantor's set theory, sparking the foundational crisis of mathematics. As a result, mathematicians developed several different flavors of set theory, among which was a well-known axiomatization of set theory by Zermelo, Fraenkel, and Skolem.

2. SETS

Roughly, a *set* is a collection of *elements*. The language of set theory contains a binary predicate symbol $\in$, called the *membership relation*. $x \in y$ means x is an element of y.

Example 2.1.

1. The empty set: $\varnothing$.
2. The set containing the empty set: $\{\varnothing\}$.
3. A set containing 3 elements: $\{a, b, c\}$.
4. The set of all natural numbers: $\mathbb{N} = \{0, 1, 2, \ldots\}$.
5. The set of all integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
6. The set containing $\mathbb{N}$ and $\mathbb{Z}$: $\{\mathbb{N}, \mathbb{Z}\}$.

Example 2.2.

1. Nothing is in the empty set: $x \notin \varnothing$.
2. The set containing the empty set has an element: $\varnothing \in \{\varnothing\}$.
3. $a \in \{a, b, c\}$, $b \in \{a, b, c\}$, $c \in \{a, b, c\}$.
4. $\mathbb{N} \in \{\mathbb{N}, \mathbb{Z}\}$, $\mathbb{Z} \in \{\mathbb{N}, \mathbb{Z}\}$, but $0 \notin \{\mathbb{N}, \mathbb{Z}\}$.

3. SUBSETS

Definition 3.1. A set x is a subset of y, denoted $x \subseteq y$, if every element in x is also in y, i.e.,

$$\forall z.(z \in x \Rightarrow z \in y)$$

The relation $\subseteq$ is called *set inclusion*.

Lemma 3.2. *The empty set is a subset of any set.*

$$\forall y. \varnothing \subseteq y$$

Proof. Let y be any set. By definition, $\varnothing \subseteq y := \forall z.(z \in \varnothing \Rightarrow z \in y)$. Let z be given. Assume that $z \in \varnothing$, but this is impossible since $z \notin \varnothing$. □

Lemma 3.3. *Every set is a subset of itself.*

$$\forall x. x \subseteq x$$

Proof. Exercise. □

Lemma 3.4. *The subset relation is transitive, i.e.,*

$$\forall x.\forall y.\forall z.\ x \subseteq y \Rightarrow y \subseteq z \Rightarrow x \subseteq z$$

Proof. Exercise. □

Example 3.5.

1. $\varnothing$ has a subset $\varnothing$.
2. $\{\varnothing\}$ has subsets $\varnothing$ and $\{\varnothing\}$.
3. $\{a, b\}$ has subsets $\varnothing$, $\{a\}$, $\{b\}$, and $\{a, b\}$.
4. $\{a, b, c\}$ has subsets $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$.

4. EQUALITY

Two sets are equal when they contain the same elements. We can express this in terms of the set inclusion relation.

$$\forall x.\forall y.\ x \subseteq y \Rightarrow y \subseteq x \Rightarrow x = y$$

Given two sets x and y, to prove that $x = y$, it suffices to prove $x \subseteq y$ and $y \subseteq x$.

Example 4.1.

1. $\{a, b, c, d, d\} = \{a, b, c, d\}$.
2. $\{a, b, c\} = \{c, b, a\}$.

5. COMPREHENSION

Given a set w, there is a subset of w whose elements satisfy a given property φ.

$$\{x \in w \mid \varphi(x)\}$$

Example 5.1.

1. The set of all even natural numbers: $\{x \in \mathbb{N} \mid \text{even}(x)\}$.
2. The set of all odd natural numbers: $\{x \in \mathbb{N} \mid \text{odd}(x)\}$.
3. The set of all integers divisible by 2: $\{x \in \mathbb{Z} \mid x \equiv 0 \mod 2\}$.
4. The set of all real numbers between 0 and 1 (inclusive): $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.

6. POWER SET

In Example 6.2, $\{a, b, c\}$ has subsets $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$. These subsets form a set

$$\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Definition 6.1. Let x be a set. The *power set* of x, denoted $\mathcal{P}(x)$, is the set of all subsets of x.

Example 6.2.

1. $\mathcal{P}(\varnothing) = \{\varnothing\}$.
2. $\mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$.
3. $\mathcal{P}(\{\varnothing, \{\varnothing\}\}) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$.
4. $\mathcal{P}(\{a, b, c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Theorem 6.3 (Cantor's Theorem)**.** *For any set x, there is no surjective function $f : x \to \mathcal{P}(x)$.*

Proof. Deferred to a later module. □

7. UNION

Definition 7.1. The *union* of two sets x and y, denoted as $x \cup y$, is a set that contains exactly those elements of x and those of y.

Example 7.2.

1. $\{1, 2, 3\} \cup \{a, b, c\} = \{1, 2, 3, a, b, c\}$.
2. $\{a, b, c\} \cup \{b, c, d\} \cup \{c, d, e\} = \{a, b, c, d, e\}$.

Remark 7.3. Set union can be characterized by a universal property: $x \cup y$ is the "smallest" set so that $x \subseteq x \cup y$ and $y \subseteq x \cup y$. That is, for any sets x, y, and z, if $x \subseteq z$ and $y \subseteq z$ then $x \cup y \subseteq z$, i.e.,

$$x \subseteq z \Rightarrow y \subseteq z \Rightarrow x \cup y \subseteq z$$

8. INTERSECTION

Definition 8.1. The *intersection* of two sets x and y, denoted as $x \cap y$, is a set that contains exactly those elements that x and y have in common.

Example 8.2.

1. $\{1, 2, 3\} \cap \{a, b, c\} = \varnothing$.
2. If $\{a, b, c\} \cap \{b, c, d\} \cap \{c, d, e\} = \{c\}$.

Remark 8.3. Set intersection can be characterized by a universal property: $x \cap y$ is the "largest" set so that $x \cap y \subseteq x$ and $x \cap y \subseteq y$. That is, for any sets x, y, and z, if $z \subseteq x$ and $z \subseteq y$ then $z \subseteq x \cap y$, i.e.,

$$z \subseteq x \Rightarrow z \subseteq y \Rightarrow z \subseteq x \cap y$$